CH5 QUANTUM GAUGE FIELDS

5.1 Gauge invariance

→ In QFT, the propagator encodes the dynamics of the fields and their interactions. However, for gauge fields, gauge invariance imposes constraints on the EOM, leading to the non-invertibility of the quadratic kernel.

Recall: quadratic kernel D, ex: $\mathcal{L}_0 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\phi D \phi$

so that $D = -\partial^2 - m^2$, or $D(p) = p^2 + m^2$

⊙ Exercice: multiple scalar fields:

→ Consider $S_0 \equiv -\frac{1}{2}\int d^n x \left(\partial_\mu \phi^i \partial^\mu \phi_i + m^2 \phi^i \phi_i\right) + i\epsilon$

$$= -\frac{1}{2}\int d^n x \int d^n x' \, \phi^i(x) \underbrace{\delta_{ij}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\mu}\delta(x-x') + m^2 - i\epsilon\right)}_{\equiv D_{ij}(x,x')} \phi^j(x')$$

→ The propagator is defined as $\Delta^{ij}(x,x') \equiv (D^{-1})^{ij}(x,x')$ so that

$$\int d^n x' \, D_{ij}(x,x')\Delta^{jk}(x',x'') = \delta_i^{\ k}\,\delta^n(x-x'')$$

↳ Since $\delta^n(x-x') = \frac{1}{(2\pi)^n}\int d^n p \, e^{ip(x-x')}$, one has

$$D_{ij}(x,x') = \delta_{ij}\frac{1}{(2\pi)^n}\int d^n p \, e^{ip(x-x')}(p^2 + m^2 - i\epsilon)$$

$$\Rightarrow \tilde{D}_{ij}(p) = \delta_{ij}(p^2+m^2-i\epsilon) \Leftrightarrow \tilde{\Delta}^{ij}(p) = \delta^{ij}\frac{1}{p^2+m^2-i\epsilon}$$

$$\Rightarrow \Delta^{ij}(x-x') = \delta^{ij}\frac{1}{(2\pi)^n}\int d^n p \, \frac{1}{p^2+m^2-i\epsilon}\, e^{ip(x-x')}$$

→ Scalar theories do not possess any gauge freedom; the field ϕ is unconstrained and all the field dof ⇔ physical dof.

→ The quadratic kernel D is fully invertible because there are no "null directions" in the functional space of the scalar fields.

Exercise: massive vector field:

→ Consider $\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}m^2 A^\mu A_\mu + i\epsilon$

$$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2}m^2 A^\mu A_\mu + i\epsilon$$

$$= -\frac{1}{2}\int d^n x\, d^n y\, A^\rho(x)\, D_{\rho\sigma}(x,y)\, A^\sigma(y)$$

with $D_{\rho\sigma}(x,y) = \left(\eta_{\rho\sigma}\frac{\partial^2}{\partial x^\mu \partial y_\mu} - \frac{\partial^2}{\partial x^\sigma \partial x^\rho} + m^2\eta_{\rho\sigma}\right)\delta^n(x-y) - i\epsilon$

$$= \frac{1}{(2\pi)^n}\int d^n p\, e^{ip(x-y)}\underbrace{\left(\eta_{\rho\sigma}(p^2+m^2) - p_\rho p_\sigma - i\epsilon\right)}_{=\tilde{D}_{\rho\sigma}}$$

→ The propagator is then given by:

$$\Delta^{\rho\sigma}(x,y) = \frac{1}{(2\pi)^n}\int d^n p\, e^{ip(x-y)}\left(\frac{\eta^{\rho\sigma} + p^\rho p^\sigma/m^2}{p^2+m^2}\right)$$

Indeed, $\left(\eta_{\rho\sigma}(p^2+m^2) - p_\rho p_\sigma\right)\left(\frac{\eta^{\sigma\tau} + p^\sigma p^\tau/m^2}{p^2+m^2}\right)$

$$= (p^2+m^2)^{-1}\left(\delta_\rho^\tau(p^2+m^2) - p_\rho p^\tau + p_\rho p^\tau(p^2+m^2)/m^2 - p^2 p_\rho p^\tau/m^2\right) = \delta_\rho^\tau$$

→ Notice that $\lim_{m\to 0} D_{\rho\sigma}$ is not invertible: $\tilde{D}_{\rho\sigma}(p) = \eta_{\rho\sigma}p^2 - p_\rho p_\sigma$

Since it's a $n\times n$ symmetric matrix, $\exists$ at least one p^σ an eigenvector of vanishing eigenvalue: $\tilde{D}_{\rho\sigma}p^\sigma = 0$

→ The quadratic part is non-invertible because of the gauge invariance. Indeed, it is equivalent to perform:

$$\int d^n y\, D_\rho{}^\sigma(x-y)\, A_\sigma^g(y) = 0$$

where $A_\sigma^g(y) \equiv \partial_\sigma \epsilon(y)$ (with $A_\nu \mapsto A_\nu + \partial_\nu \epsilon$ the gauge freedom)

A pure gauge field: $A_\sigma = A_\sigma^{phys} + A_\sigma^g = A_\sigma^{phys} + \partial_\sigma \epsilon = \partial_\sigma \epsilon$ annihilates the quadratic part; the pure gauge modes lie in the null space of the propagator due to gauge invariance.

5.2 Invariance BRST

→ Consider a gauge invariant action $S^{inv}[A, y^i] = \int d^n x\, L^{inv}[A, y^i]$ where $y^i \equiv (\phi, \xi, \psi)$ are the matter fields : scalar, Weyl fermions and Dirac fermions.

⊙ Yang-Mills theory:

DEF The Yang-Mills lagrangian L^{YM} is defined as
$$L^{YM} \equiv \frac{-1}{4g^2} F_{\mu\nu}{}^a F^{\mu\nu b} g_{ab} \quad (\text{with } f^d{}_{ac}\, g_{db} + f^d{}_{bc}\, g_{da} = 0)$$

where g_{ab} is the Killing metric of the group, $g_{ab} \propto \mathrm{Tr}\{ad_a \circ ad_b\}$
and where $F_{\mu\nu}{}^a \equiv \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f^a{}_{bc} A_\mu{}^b A_\nu{}^c$
$f^a{}_{bc}$ are the structure constants of the gauge group $[T_a, T_b] = i f_{ab}^c T_c$

→ Consider $L^{inv}[A, y^i] = L^{YM} + L_M[y^i; Dy^i]$, with the following infinitesimal gauge transformations :
$$\delta_\varepsilon S^{inv} = 0 \Rightarrow \begin{cases} \delta_\varepsilon A_\mu{}^a = D_\mu \varepsilon^a = \partial_\mu \varepsilon^a + g f^a{}_{bc} A_\mu{}^b \varepsilon^c \\ \delta_\varepsilon y^i = -\varepsilon^a (T_a)^i{}_j\, y^j \end{cases}$$
where $\varepsilon^a(x)$ is an arbitrary field.

→ If we rescale the gauge field $A_\mu^a \mapsto g A_\mu^a$, the structure constants $f^{abc} \mapsto g f^{abc}$ and the generator of the gauge group : $T_a^i \mapsto g T_a^i$, we get the canonical lagrangian normalization

⊙ Chern-Simons theory:

DEF The Chern-Simons lagrangian L^{CS} in 3D is defined as:
$$L^{CS} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} g_{ab} A_\mu^a \left(\partial_\nu A_\rho{}^b + \frac{1}{3} f^b{}_{cd} A_\nu^c A_\rho^d\right)$$

where k is the Chern-Simons level. It defines a topological field theory in 3-D of spacetime.

DEF The gauge field connection A is defined as

$$A = A^a_\mu \, dx^\mu \, T_a$$

→ The gauge field is a Lie algebra-valued 1-form

The inner product on the Lie algebra is

$$\langle T_a, T_b \rangle = g_{ab}$$

The exterior derivative d of a 1-form is given by

$$dA = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu \quad \text{where } A_\mu = A^a_\mu T_a$$

The graded commutator $[A, A]$ is defined as

$$[A, A] = f_{ab}{}^c A^a \wedge A^b T_c$$

→ Notice that $F = dA + \frac{1}{2}[A, A]$. Indeed,

$$dA + \frac{1}{2}[A, A] = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu + \frac{1}{2} f_{ab}{}^c A^a \wedge A^b T_c$$

$$= \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) dx^\mu \wedge dx^\nu T_a + \frac{1}{2} f_{ab}{}^c A^a_\mu A^b_\nu \, dx^\mu \wedge dx^\nu T_c$$

$$= \frac{1}{2}\left((\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) T_a + f_{ab}{}^c A^a_\mu A^b_\nu T_c\right) dx^\mu \wedge dx^\nu$$

so that $F_{\mu\nu}{}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_a{}^{bc} A_\mu{}^b A_\nu{}^c$

→ Notice that $\langle A, dA \rangle = \langle A^a_\sigma \, dx^\sigma T_a, \partial_{(\mu} A^b_{\nu)} \, dx^\mu \wedge dx^\nu T_b \rangle$

$$= A^a_\sigma \partial_{(\mu} A^b_{\nu)} \, dx^\sigma \wedge dx^\mu \wedge dx^\nu \langle T_a, T_b \rangle = g_{ab} A^a \wedge dA^b$$

PROP The Chern-Simons action can be written as

$$S^{CS} = \frac{k}{4\pi} \int \langle A, dA \rangle + \frac{2}{3} \langle A^3 \rangle = \frac{k}{8\pi} \int \langle A, dA + \frac{2}{3} A^2 \rangle$$

$\boxed{\text{DEMO}}$ $$d^3x \, L^{CS} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \big(\underbrace{g_{ab} A^a_\mu \partial_\nu A^b_\rho}_{= 2\langle A, dA \rangle} + \frac{1}{3} g_{ad} f^{bcd} A^a_\mu A^c_\nu A^d_\rho \big) d^3x$$

Now, $\langle A^3 \rangle = \langle A \wedge A \wedge A \rangle = \langle A^a_\mu A^b_\nu A^c_\rho \, dx^\mu \wedge dx^\nu \wedge dx^\rho \, T_a T_b T_c \rangle$

and $T_{[a} T_b T_{c]} = \frac{1}{3!} f^{abc} T_c$ so that $\langle A^3 \rangle = g_{ad} f^{dbc} A^a \wedge A^b \wedge A^c$ ∎

→ On can normalize $A^a{}_\mu \mapsto g A^a{}_\mu$, $f^a{}_{bc} \mapsto g f^a{}_{bc}$ so that the quadratic part of L^{cs} becomes $L^{CS}_{quad} = \frac{1}{2} \epsilon^{\mu\nu\alpha} A^a_\mu \partial_\nu A^b_\alpha g_{ab}$

→ The gauge transformation $\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a$ leaves L^{CS}_{quad} invariant up to a total derivative term → topological.

⊙ ghosts:

→ We replace the gauge parameter $\varepsilon^a(x)$ with a ghost field $C^a(x)$

$$\varepsilon^a(x) \to C^a(x)$$

→ These field are fermionic (anticomuting) scalars hence they violate the spin-statistics theorem. It's ok since they're not irrep of the Poincaré group.

→ The fields in the theory are now:

$\phi^A \equiv (A^a_\mu, y^i, C^a, \bar{C}^a, B^a)$ fields

$\phi^{*A} \equiv (A^{*\mu}_a, y^*_i, C^*_a, \bar{C}^*_a, B^*_a)$ conjugate fields (antifield)

with B^a a auxiliary bosonic field.

DEF We introduce the parity p that indicates if a field is comuting ($p=0$) or anticomuting ($p=1$), and the ghost number gh that indicates if the field is a ghost ($gh=1$), an antighost ($gh=-1$) or neither ($gh=0$)

For the antifields, we have the following rules:

$$p(\phi^*_A) = p(\phi^A) + 1 \pmod 2 \qquad gh(\phi^*_A) = -gh(\phi^A) - 1$$

→ One has

	A^a_μ	ϕ	ξ	ψ	C^a	$\bar{C}^a$	B^a
p	0	0	1	1	1	1	0
gh	0	0	0	0	1	-1	0
	$A^{*\mu}_a$	ϕ^*	ξ^*	ψ^*	C^*_a	$\bar{C}^*_a$	B^*_a
p	1	1	0	0	0	0	1
gh	-1	-1	-1	-1	-2	0	-1

→ Next, we introduce a graded analog of the Poisson bracket used in the Batalin-Vilkovsky formalism (BV).

eq. 17.26 Henneaux

DEF The antibracket $(\,,\,)$ is defined for two functionals F and G:

$$(F,G)=\int d^nx \left\{ \frac{\delta^R F}{\delta\phi^A(x)}\frac{\delta^L G}{\delta\phi^*_A(x)} - \frac{\delta^R F}{\delta\phi^*_A(x)}\frac{\delta^L G}{\delta\phi^A} \right\}$$

with the right derivative and left derivative such that

$$\int d^nx\, \delta\phi^B(x)\frac{\delta^L F}{\delta\phi^B(x)} = \int d^nx\, \frac{\delta^R F}{\delta\phi^B(x)}\delta\phi^B(x)$$

$$\Leftrightarrow \frac{\delta^R F}{\delta\phi^B(x)} = (-1)^{B(B+F)}\frac{\delta F}{\delta\phi^B(x)} \quad \text{with} \begin{cases} B \text{ the parity of } \phi^B(x) \\ F \text{ the parity of } F[\phi^A] \end{cases}$$

PROP ① Graded antisymmetry:

$$(F,G) = -(-1)^{(F+1)(G+1)}(G,F)$$

② Graded Jacobi identity

$$(F,(G,H)) = ((F,G),H) + (-1)^{(F+1)(G+1)}(G,(F,H))$$

$$\Leftrightarrow (F,(G,H)).(-1)^{(F+1)(H+1)} + \text{cyclic}(F,G,H) = 0$$

③ if $(-1)^F = +1$, then $\frac{1}{2}(F,F) = \int d^nx \frac{\delta^R F}{\delta\phi^A(x)}\frac{\delta^L F}{\delta\phi^*_A(x)} = -\int d^nx \frac{\delta^R F}{\delta\phi^*_A(x)}\frac{\delta^L F}{\delta\phi^A(x)}$

so $(F,F)\neq 0$ but $(F,(F,F))=0$ from ②

⊙ Master action and BRST transformations:

DEF The master action S is given by

$$S[\phi^A,\phi^*_A] = S^{inv} + \int d^nx \left(-D_\mu C^a A^{*\mu}_a + C^a (T_a)^i{}_j y^j y^*_i + \frac{1}{2} f^a{}_{bc} C^b C^c C^*_a - B^a \bar{C}^*_a\right)$$

We denote the BRST differential as $s \equiv (S,\cdot)$
↳ Becchi - Rouet - Stora - Tyutin

→ For instance, $s\phi^A(x) = (S,\phi^A) = -\frac{\delta^R S}{\delta\phi^*_A(x)}$ and $s\phi^*_A(x) = \frac{\delta^R S}{\delta\phi^A(x)} = (-1)^A\frac{\delta^L S}{\delta\phi^A(x)}$

→ One sees that

$$\begin{cases} sA^a_\mu = D_\mu C^a \qquad s y^i = -C^a (T_a)^i{}_j y^j \qquad s C^a = -\frac{1}{2} f^a{}_{bc} C^b C^c \\ s\bar{C}^a = B^a \qquad sB^a = 0 \end{cases}$$

PROP On the fields A^a_μ, y^i, the BRST differential (or BRST transformation) act like gauge transformations with $\varepsilon^a(x) \mapsto C^a(x)$

→ Explicitely, the covariant derivative in the adjoint rep. of the gauge group is

$$D_\mu v_a = \partial_\mu v_a - f_a{}^b{}_c A^c_\mu v_b \quad \text{for } v_a \text{ any vector in the Lie alg.}$$

For example: $D_\mu C^a A^{*\mu}_a = \partial_\mu C^a A^{*\mu}_a + f_b{}^a{}_c A^b_\mu C^c A^{*\mu}_a$

→ Let us compute the various eq. of motions $\frac{\delta S}{\delta \phi^A} = 0$, with left derivatives:

$$\begin{cases} \dfrac{\delta^L S}{\delta A^a_\mu} = \dfrac{\delta S^{inv}}{\delta A^a_\mu} - f_a{}^b{}_c C^c A^{*\mu}_a & \dfrac{\delta^L S}{\delta \bar{C}^a} = 0 \\ \dfrac{\delta^L S}{\delta y^i} = \dfrac{\delta^L S^{inv}}{\delta y^i} + (-1)^i C^a (T_a)^j{}_i y^*_j & \dfrac{\delta^L S}{\delta B^a} = -\bar{C}^*_a \\ \dfrac{\delta^L S}{\delta C^a} = D_\mu A^{*\mu}_a + (T_a)^i{}_j y^j y^*_i + C^*_c f_a{}^c{}_b C^b & \end{cases}$$

PROP | The ghost number of the master action and of the BRST diff. are

$$gh(S) = 0 \qquad \text{and} \qquad gh(s) = 1$$

PROP | The classical master equation reads

$$\boxed{(S,S) = 0 \quad \Leftrightarrow \quad sS = 0}$$

→ The master eq. shows the BRST invariance of the master action.

DEMO | One has:

$$\frac{1}{2}(S,S) = \int d^n x \left\{ s\phi^A \cdot \frac{\delta^L S}{\delta \phi^A} \right\} = \int d^n x \left\{ \left(-\frac{\delta^R S}{\delta \phi^*_A} \right) \cdot \frac{\delta^L S}{\delta \phi^A} \right\}$$

$$= \int d^n x \left\{ -\frac{\delta^R S}{\delta A^*_\mu} \cdot \frac{\delta^L S}{\delta A^\mu} - \frac{\delta^R S}{\delta y^*_i} \cdot \frac{\delta^L S}{\delta y^i} - \frac{\delta^R S}{\delta C^*_a} \cdot \frac{\delta^L S}{\delta C^a} - \underbrace{\frac{\delta^R S}{\delta \bar{C}^*_a} \frac{\delta^L S}{\delta \bar{C}^a}}_{0} - \underbrace{\frac{\delta^R S}{\delta B^*_a} \frac{\delta^L S}{\delta B^a}}_{0} \right\}$$

$$= \int d^n x \Big\{ D_\mu C^a \Big(\frac{\delta S^{inv}}{\delta A^a_\mu} - \underbrace{f_a{}^b{}_c C^c A^{*\mu}_a}_{1} \Big) - C^a (T_a)^i{}_j y^j \Big(\frac{\delta S^{inv}}{\delta y^i} + (-1)^k (T_b)^l{}_k y^*_l \Big)$$

$$- \frac{1}{2} f_b{}^a{}_c C^b C^c \Big(\underbrace{\partial_\mu A^{*\mu}_a - f_e{}^d{}_a A^e_\mu A^{*\mu}_d}_{1} + (T_a)^i{}_j y^j y^*_i + C^*_c f_a{}^c{}_e C^e \Big) \Big\}$$

Now, by definition, S^{inv} is gauge invariant so that $\delta_A S^{inv} = 0 = \delta_y S$

$$1 = -\partial_\mu C^a \cdot f_a{}^b{}_c C^c A^{*\mu}_b - f_d{}^a{}_e A^d_\mu C^e f_a{}^b{}_c C^c A^{*\mu}_b$$
$$- \frac{1}{2} f_b{}^a{}_c C^b C^c \cdot \partial_\mu A^{*\mu}_a + \frac{1}{2} f_b{}^a{}_c C^b C^c f_c{}^d{}_a A^e_\mu A^{*\mu}_d$$
$$= -\partial_\mu \left(\frac{1}{2} f_a{}^b{}_c C^a C^c A^{*\mu}_b \right) + \frac{1}{2} C^b C^c \begin{pmatrix} f_e{}^d{}_a f_b{}^a{}_c - f_b{}^d{}_c f_e{}^a{}_b + f_a{}^d{}_b f_e{}^a{}_c \\ + f_c{}^d{}_a f_e{}^a{}_b + f_b{}^d{}_a f_c{}^a{}_e \end{pmatrix} A^e_\mu A^{*\mu}_d$$
$$= \text{boundary term} \to 0$$

We're left with:

$$\frac{1}{2}(S,S)=\int d^nx\Big\{-C^a(T_a)^i{}_j\, y^j\,.(-1)^k C^b(T_b)^l{}_k\, y^*_l$$
$$-\frac{1}{2}f_b{}^a{}_c\, C^bC^c\big((T_a)^i{}_j\, y^j y^*_i+\underbrace{C^*_c f_a{}^c{}_e\, C^e}\big)\Big\}$$

$\to$ C's contracted with $e,b,c\to$ anticommutation on those indices

$\uparrow = -\frac{1}{2}C^*_c\, f_{a[e}\, f_b{}^a{}_{c]}\, C^bC^cC^e=0$ (Jacobi)

$\to$ Recall that $\left(\frac{1}{2}[T_a,T_b]\right)^k{}_j=+\frac{1}{2}f_b{}^c{}_a(T_c)^k{}_j$. We're left with

$$\frac{1}{2}(S,S)=\int d^nx\Big\{-C^a(T_a)^i{}_j\,y^j(-1)^i C^b(T_b)^k{}_i\,y^*_k-\frac{1}{2}f_b{}^a{}_c\,C^bC^c(T_a)^i{}_j\,y^jy^*_i\Big\}$$

$$=\int d^nx\Big\{-C^aC^b(T_b)^k{}_i(T_a)^i{}_j\,y^jy^*_k(-1)^i-\frac{1}{2}f_b{}^a{}_c\,C^bC^c(T_a)^i{}_j\,y^jy^*_i$$

$$=\int d^nx\Big\{-\frac{1}{2}C^aC^b([T_a,T_b])^k{}_j\,y^jy^*_k-\frac{1}{2}f_b{}^a{}_c\,C^bC^c(T_a)^i{}_j\,y^jy^*_i\Big\}=0 \quad \blacksquare$$

Corr The BRST operator s is nilpotent:

$$s^2=0\iff s^2\phi^A=0\ \forall\phi^A$$

$\boxed{\text{DEMO}}$ One has:

$$s^2F=(S,(S,F))=((S,S),F)+(-1)^{(P_S+1)(P_S+1)}(S,(S,F))\ \text{ and } P_S=0$$
$$=((S,S),F)-(S,(S,F))$$
$$\iff (S,(S,F))=\frac{1}{2}((S,S),F)=0 \quad \blacksquare$$

$\to$ ex: $sA^a_\mu=D_\mu C^a$ and

$$s^2A^a_\mu=sD_\mu C^a=s(\partial_\mu C^a+f_b{}^a{}_c\,A^b_\mu C^c)$$
$$=\partial_\mu sC^a+sf_b{}^a{}_c\,A^b_\mu C^c$$
$$=\partial_\mu\left(-\frac{1}{2}f_b{}^a{}_c\,C^bC^c\right)+f_b{}^a{}_c\,D_\mu C^bC^c-f_b{}^a{}_c\,A^b_\mu\,.\,\frac{1}{2}f_d{}^c{}_g\,C^dC^g=0$$

$\to$ The antifield formalism is usefull when dealing with theories that are not Yang-Mills, when the Fadeev-Popov Procedure is no longer valid. The antifield are part of the Batalin-Vilkovisky framework.

5.3 Gauge fixation and propagators

a) Canonical transfo and generating function of the 2^{nd} kind:

DEF Consider q a generalized coord. and p its conjugated momentum: $p = \partial_{\dot q} L$.
They satisfy: $\dot q = \partial_p H$ and $\dot p = -\partial_q H$. Writing the Poisson bracket

$$\{F, G\} = \sum_a \left(\frac{\partial F}{\partial q^a}\frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a}\frac{\partial G}{\partial q^a} \right)$$

one has $\{q^a, p_a\} = \delta^a_b$; $\{q^a, q^b\} = 0$, $\{p_a, p_b\} = 0$.
A canonical transformation $(Q(q,p), P(q,p))$ is one that preserve
$\{Q(q,p), P(q,p)\} = 1$ and $\{Q,Q\} = 0 = \{P,P\}$
Equivalently, for $\tilde{G}(Q,P) = G(q(Q,P), p(Q,P))$, one has
$\{\tilde{G}_1, \tilde{G}_2\}_{Q,P} = \{G_1, G_2\}_{q,p}$

→ Canonical transfo. can be defined using generating functions F. There are 4 types: $F_1(q,Q)$, $F_2(q,P)$, $F_3(p,Q)$, $F_4(p,P)$.
For a generating function of the 1^{nd} kind:
we write $p\,dq = P\,dQ + dF_1(q,Q)$ so that
$p = \frac{\partial F_1}{\partial q}$ and $P = -\frac{\partial F_1}{\partial Q}$. Here, the independent variables are (q,Q).

Then, we integrate to get q and P:

$$p\,dQ = -dP.Q + d\underbrace{\left(F_1(q,P) + PQ\right)}_{= F_2(q,P) = qP + \Psi(q)}$$

The induced canonical transformations are:

$p = P + \frac{\partial \Psi}{\partial q}$ and $q = Q$ shift in momentum, same position

→ We can generalize it to QFTs: $(q,p) \mapsto (\phi(x), \pi(x))$, with
$\{\phi(x), \pi(x)\} = \delta(x,y)$ and $\{\phi,\phi\} = 0 = \{\pi,\pi\}$
The generating functional becomes $F_2[\phi(x), \Pi(x)]$ with
$\pi(x) = \frac{\delta F_2}{\delta \phi(x)}$ and $\Phi(x) = \frac{\delta F_2}{\delta \Pi(x)}$ // $\partial_q F_2 = p$ and $\partial_P F_2 = Q$

Picking $F_2[\phi, \Pi] = \int dx\, \phi(x)\Pi(x) + \Psi[\phi]$, one finds
$\pi(x) = \Pi(x) + \frac{\delta \Psi[\phi]}{\delta \phi(x)}$ and $\Phi(x) = \phi(x)$

⊙ Generalization to the antifield formalism:

→ We consider the action $S[\phi^A, \phi^*_A]$, so that $(q,p) \mapsto (\phi^A, \phi^*_A)$.

DEF: An anticanonical transformation in this formalism preserves the antibrackets: $(\phi^A, \phi^*_A) = (\tilde{\phi}^A, \tilde{\phi}^*_A)$

→ We consider $F[\phi^A, \tilde{\phi}^*_A] = \int d^n x \{ \overset{q}{\phi_A(x)} \overset{P}{\tilde{\phi}^*_A(x)} - \overset{P}{\tilde{\phi}^A(x)} \overset{Q}{\tilde{\phi}^*_A(x)} + \overset{\Psi}{\Psi(\phi)} \}$

Indeed, $p\,dq = P\,dQ + dF$, translates to
$\int d^n x\, \delta\phi^A . \phi^*_A = \int d^n x\, \delta\phi^A . \tilde{\phi}^*_A + \delta F$
with $F = \int d^n x (\tilde{\phi}^*_A \phi^A - \tilde{\phi}^A \tilde{\phi}^*_A + \Psi)$

One finds:
$$\phi^A = \tilde{\phi}^A \quad \text{and} \quad \phi^*_A = \tilde{\phi}^*_A + \frac{\delta^L \Psi}{\delta\phi^A(x)}$$

DEF: In our case, we consider the following gauge-fixing functional
$$\Psi = \int d^n x\, \bar{C}^a (\partial_\mu A^{b\mu} - \frac{\xi}{2} B^b) g_{ab}$$
This defines a class of gauge-fixing condition, with the gauge-fixing parameter ξ. This is refered as R_ξ gauges.
→ The Landau gauge is $\xi = 0$
→ The Feynman gauge is $\xi = 1$

→ The gauge-fixed action S_Ψ reads:
$$S_\Psi[\tilde{\phi}, \tilde{\phi}^*] = S\left[\phi, \tilde{\phi}^* + \frac{\delta^L\Psi}{\delta\phi}\right] = S^{inv}[\phi] - \int d^n x\, s\phi^A(x) \left(\tilde{\phi}^*_A + \frac{\delta^L\Psi}{\delta\phi^A}\right)$$
$$= S^{inv}[\phi] - \int d^n x\, s\phi^A \tilde{\phi}^*_A - s\Psi$$
with
$$-s\Psi = \int d^n x \{ \underbrace{-\partial^\mu \bar{C}^a D_\mu C^b}_{\text{kinetic term for the ghosts}} \underbrace{- B^a \partial^\mu A^b_\mu}_{\text{gauge fixing for } A^a_\mu} + \underbrace{\frac{\xi}{2} B^a B^b}_{\to \text{auxiliary fields}} \} g_{ab}$$

→ We have 2 options:
1) Integrate out B_a:
→ solve $\frac{\delta S_\Psi}{\delta B^a} = 0$, and find $B_a = \frac{1}{\xi}(\partial_\mu A^{a\mu} + \tilde{\bar{C}}^a)$

→ The integral becomes a gaussian.

$$S_\Psi\Big|_{B_a} = S^{inv} + \int d^n x \left\{ -\partial^\mu \bar{C}^a D_\mu C^b - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu + \tilde{\bar{C}}^{*a}\right)\left(\partial^\nu A^b_\nu + \tilde{\bar{C}}^{*b}\right)\right\} g_{ab}$$
$$- \int d^n x \left\{ s A^a_\mu \tilde{A}^{*\mu}_a + s C^a \tilde{C}^*_a + s y^i \tilde{y}^*_i \right\}$$

→ In both option, the action remains BRST-invariant:

$$\frac{1}{2}(S,S) = 0 = \frac{1}{2}(S_\Psi, S_\Psi)_{\phi,\tilde{\phi}^*} = 0 \iff s_\Psi S_\Psi[\phi, \tilde{\phi}^*] = 0$$

Indeed, $s_\Psi = (S_\Psi, \cdot)_{\phi,\tilde{\phi}^*}$ and $s_\Psi \phi^A = (S_\Psi, \phi^A)_{\phi,\tilde{\phi}^*} = s\phi^A$

while $s_\Psi \tilde{\phi}^*_A = (S_\Psi, \tilde{\phi}^*_A) = \dfrac{\delta^L (S^{inv} - s\Psi - \int d^n x\, s\phi^B \cdot \tilde{\phi}^*_B)}{\delta \phi^A}$

2) Keep B_a in the action:

→ Easier to compute the propagators.

⊙ Gauge field propagator with auxiliary field integrated out: YM

→ Lets compute the propagator in a YM theory using $S_\Psi\big|_B$ (B_a gone).

The quadratic part of the action reads:

$$S_\Psi\Big|^{(2)}[\phi, 0] = \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu b} - \frac{1}{2\xi} \partial^\mu A^a_\mu \partial^\nu A^b_\nu - \partial_\mu \bar{C}^a \partial^\mu \bar{C}^b \right\} g_{ab}$$

Choosing the normalization of the generators of $SU(N)$ to be $Tr\{\lambda_a \lambda_b\} = \delta_{ab}$, we have $g_{ab} = \delta_{ab}$.

$$S_\Psi\Big|^{(2)} = \int d^4 x \left\{ -\frac{1}{4} \underbrace{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b})}_{= 2 A^a_\mu (\eta^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A^b_\nu} - \underbrace{\frac{1}{2\xi}(\partial_\mu A^{\mu a})^2}_{= \frac{-1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^b_\nu} - (\partial_\mu \bar{C}^a)^2 \right\} \delta_{ab}$$

$$= -\frac{1}{2} \phi^A_{(B)} D^{(B)}_{AB} \phi^B_{(B)} - \bar{\Phi}^\kappa_{(F)} D^{(F)}_{\kappa\beta} \phi^\beta_{(F)} \quad \text{with} \begin{cases} (B) \text{ for bosons} \\ (F) \text{ for fermions} \end{cases}$$

$$= -\frac{1}{2} \int d^4 x \int d^4 y \, D^{\mu\nu}_{ab}(x,y) A^a_\mu(x) A^b_\nu(y) - \int d^4 x \int d^4 y \, \bar{C}^a(x) D^{gh}_{ab}(x,y) C^b(y)$$

where the kernels $D^{\mu\nu}_{ab}(x,y)$ and $D^{gh}_{ab}(x,y)$ are given by

$$D^{\mu\nu}_{ab}(x,y) = \left(\eta^{\mu\nu} \frac{\partial^2}{\partial x^\sigma \partial x_\sigma} \delta^4(x-y) - \left(1 - \frac{1}{\xi}\right) \frac{\partial^2}{\partial x_\mu \partial x_\nu} \delta^4(x-y) \right) g_{ab} + \mathcal{O}(\epsilon)$$

$$= (2\pi)^{-4} \int d^4 p \left(\eta^{\mu\nu}(p^2 - i\epsilon) - (1 - 1/\xi) p_\mu p_\nu \right) e^{ip(x-y)} \cdot g_{ab}$$

The propagator is:

$$(D^{-1})^{ab}_{\mu\nu} = (2\pi)^{-4} \int d^4 p \left(\eta_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 - i\epsilon} e^{ip.(x-y)} g^{ab}$$

→ In the Landau gauge ($\xi = 0$), one has a simpler propagator but the kernel is ill-defined.
The simplest propagator is in the Feynman gauge ($\xi = 1$), but it requin a IR cut off on top of the UV one.

→ The kernel for the ghosts $-\int d^4x \int d^4y\, \bar{c}^a(x)\, D^{gh}_{ab}\, \bar{c}^b(y)$ is
$$D^{gh}_{ab}(x,y) = \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x_\sigma}\delta^4(x-y)\, g_{ab} = (2\pi)^{-4}\int d^4p\,(p^2 - i\epsilon)\, e^{ip(x-y)}\, g_{ab}$$
The propagator is then
$$(D^{-1}_{gh})^{ab} = \frac{1}{(2\pi)^4}\int d^4p\, \frac{g^{ab}}{p^2 - i\epsilon}\, e^{ip(x-y)}$$

⊙ Gauge field propagator keeping auxiliary fields : YM

→ Recall we have $S_\Phi = S^{inv} - \int d^nx\; s\phi^A \cdot \phi^*_A - s\Phi$ with
$$-s\Phi = \int d^4x\,(-\partial^\mu \bar{c}^a \partial_\mu c^b - B^a \partial^\mu A^b_\mu + \xi/2 \,.\, B^a B^b)\, g_{ab}$$
→ We want to write $S_\Phi^{(2)}$ like $S_\Phi^{(2)} = -\frac{1}{2}\phi^A_{(B)} D^{(B)}_{AB}\phi^B_{(B)} - \bar{\Phi}^\alpha_{(F)} D^{(F)}_{\alpha\beta}\phi^\beta_{(F)}$
→ We have 5 bosonic fields : A_μ, B^b. (since $g_{ab} = \delta_{ab}$, $\{B^a\} = 1$ field)
The ghost fields are decoupled, so we don't discussed them.
The kernel of $S_\Phi^{(2)}$ reads
$$-\frac{1}{2}\big(A_\mu(x), B(y)\big)\begin{pmatrix} D^{\mu\nu}_{ab}(x,y) & D^\mu_{ab}(x,y) \\ D_{ab}{}^\nu(x,y) & D_{ab}(x,y)\end{pmatrix}\begin{pmatrix} A^\nu(x) \\ B(y)\end{pmatrix}$$
→ Since $S^{(2)}_{\Phi,B} = -\frac{1}{2}\int d^nx\, \frac{\xi}{2} B^a B^b g_{ab}$, one finds $D_{ab}(x,y) = -\xi\, g_{ab}$
→ We already found:
$$D^{\mu\nu}_{ab}(x,y) = \left(\eta^{\mu\nu}\frac{\partial^2}{\partial x^\sigma \partial x_\sigma}\delta^4(x-y) - \frac{\partial^2}{\partial x^\mu \partial x^\nu}\delta^4(x-y)\right) g_{ab}$$
→ The non diagonal terms comes from $S^{(2)}_{\Phi,BA} = \int d^nx\, B^a(x)\,\partial^\mu A^a_\mu(x)$
$\Leftrightarrow S^{(2)}_{\Phi,BA} = \int d^nx \int d^ny\, B^a(x)\left(\frac{\partial}{\partial x_\mu}\delta^4(x-y)\right) A^a_\mu(y)$

so that $D_{ab}{}^\mu(x,y) = -g_{ab}\frac{\partial}{\partial x_\mu}\delta^4(x-y)$; $D^\nu_{ab}(x,y) = -g_{ab}\frac{\partial}{\partial y_\nu}\delta^4(x-y)$

→ The Fourier matrix reads:
$$\tilde{D}(p) = g_{ab}\begin{pmatrix} \eta^{\mu\nu}p^2 - p^\mu p^\nu & -ip^\mu \\ ip^\nu & -\xi \end{pmatrix}$$

→ The inverse matrix is

$$g^{bc}\begin{pmatrix} \dfrac{\eta_{\nu\rho} + (\xi-1)\, p_\nu p_\rho/p^2}{p^2} & -i\,\dfrac{p_\nu}{p^2} \\ i\,\dfrac{p_\rho}{p^2} & 0 \end{pmatrix} = \tilde{D}^{-1}$$

⊙ Gauge field propagator keeping auxiliary field: Chern-Simons

→ The ghost and auxiliary part of the action are unchanged. Instead for considering $S^{inv(2)} = \int d^n x\, F^a_{\mu\nu} F^{\mu\nu b} g_{ab}\big|^{(2)}$, we consider

$$S_{CS}^{inv(2)} = \frac{1}{2}\int d^3x\, \epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^b_\rho\, g_{ab} \quad \text{(canonical normalization)}$$

$$= -\frac{1}{2}\int d^3x \int d^3y\, A^a_\mu(x)\, D^{\mu\nu}_{ab}(x,y)\, A^b_\nu(y)$$

with $D^{\mu\nu}_{ab}(x,y) = \epsilon^{\mu\nu\rho}\dfrac{\partial}{\partial y^\rho}\delta^3(x-y)\, g_{ab} = (2\pi)^{-3}\int d^3p\,(-i\epsilon^{\mu\rho\nu}p_\rho)\, g_{ab}\, e^{ip(x-y)}$

→ The kernel of the quadratic part reads

$$\tilde{D}_{ab}(p) = g_{ab}\begin{pmatrix} i\epsilon^{\mu\nu\rho}p_\rho & -ip^\mu \\ ip^\nu & -\xi \end{pmatrix}$$

and the propagator is given by

$$(\tilde{D}^{-1})^{bc}(p) = \begin{pmatrix} \dfrac{(-1)^s i\epsilon_{\nu\lambda\sigma}p^\sigma + p_\nu p_\lambda \xi/p^2}{p^2} & -i\,\dfrac{p_\nu}{p^2} \\ i\,\dfrac{p_\lambda}{p^2} & 0 \end{pmatrix} g^{bc}$$

where the signature of the metric s comes from $\epsilon^{\lambda\mu\nu}\epsilon_{\lambda\kappa\rho} = (-1)^s(\delta^\mu_\kappa\delta^\nu_\rho - \delta^\mu_\rho\delta^\nu_\kappa)$

→ Notice that $(D^{-1})_{ab}$ is always 0: the auxiliary field B^a does not propagate as an independent dof.

→ Here, the Landau gauge will give the simplest propagator.

5.4 Vanishing of Chern-Simons β-function

→ To acces the β-function, one needs to control the renormalization of the coupling constant g. To do so, one can use the effective action $\Gamma[\bar{A}]$ with $\bar{A}$ the constant classical field:

$$\Gamma[\bar{A}] = -V_{eff}[\bar{A}] \int d^3x = -V_{eff}[\bar{A}]\,(2\pi)^3 \delta^3(0)$$ // p 42 here

→ For the scalar field with found (p42) that ($i\, g\phi^4/4!$):

$$\Gamma[\Phi] = S[\Phi] - \frac{\hbar}{2i} \mathrm{Tr}\left\{ \ln\left[\delta^4(x-y) + D^{-1}(x,y)\, \frac{g}{2}\Phi^2(y) \right] \right\} + \mathcal{O}(\hbar^2)$$

When developping $\ln(1+x) = \sum (-1)^n x^n/n$, we found that

$$\Gamma[\Phi] = S[\Phi] + \frac{\hbar}{2i} \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln\left[1 + \frac{g\Phi^2/2}{p^2+m^2-i\epsilon} \right]$$

→ The 1-loop effective potential of CS is given by

$$V_{eff}^{(1)}[\bar{A}] = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2i} \mathrm{Tr}\left\{ \ln\left[\tilde{D}^{-1}_{(B)}(p)\, \tilde{D}^{\bar{A}}_{(B)}(p) \right] \right\} - \frac{1}{i} \mathrm{Tr}\left\{ \tilde{D}^{-1}_{gh}(p)\, \tilde{D}^{\bar{A}}_{gh}(p) \right\} \right\}$$

where $\tilde{D}^{\bar{A}} \equiv \left. \frac{\delta^2 S[\phi, \phi^* = 0]}{\delta\phi^A \delta\phi^B} \right|_{A=\bar{A}}$ and the trace is on μ, ν, a, b.

Not | We introduce the following notations:

$(\bar{a}_\mu)^a{}_b \equiv \bar{A}^c_\mu f_c{}^a{}_b$ and $p_\mu{}^a{}_b \equiv \delta^a_b p_\mu + i\,\bar{a}_\mu{}^a{}_b$

$p\,\bar{a} \equiv p^\mu (\bar{a}_\nu)^a{}_b = p^\mu \bar{A}^c_\nu f_c{}^a{}_b$

and $p\bullet\bar{a} \equiv p^\sigma (\bar{a}_\sigma)^a{}_b$, and $(p\bullet\bar{a})\delta = p^\sigma (\bar{a}_\sigma)^a{}_b \delta^\mu_\nu$

→ We these notations, and setting $\xi = 0$, the quadratic part becomes

$$\tilde{D}^{\bar{A}}_{bos} = g^{ab} \begin{pmatrix} i\epsilon^{\mu\nu\rho} p_\rho & -ip^\mu \\ ip^\nu & -\xi \end{pmatrix} = \begin{pmatrix} i\epsilon^{\lambda\varepsilon\rho} P_\rho & -i\delta p^\lambda \\ i\delta p^\varepsilon & 0 \end{pmatrix}$$ and

$$(\tilde{D}^{-1}_{bos})^a{}_b = \delta^a{}_b \begin{pmatrix} i(-1)^s \epsilon_{\nu\lambda\sigma} p^\sigma/p^2 & -ip_\nu/p^2 \\ ip_\lambda/p^2 & 0 \end{pmatrix}$$

so that:

$$\tilde{D}^{-1}_{bos} \tilde{D}^{\bar{A}}_{bos} = \left(\begin{array}{c|c} \delta\delta + i\,\frac{p\bullet\bar{a}}{p^2}\delta - i\,\frac{p\,\bar{a}}{p^2} & 0 \\ \hline -i\,\frac{p_\lambda}{p^2}\epsilon^{\lambda\varepsilon\rho}\bar{a}_\rho & \delta \end{array} \right)$$

→ For the ghost, one has $\tilde{D}_{gh}^{\bar{A}} = \delta_{ab}\, p^2 + i p\cdot\bar{a}$
and $\tilde{D}_{gh}^{-1} = \delta^{ab}\, 1/p^2$

↳ One finds $\tilde{D}_{gh}^{-1}\tilde{D}_{gh}^{\bar{A}} = \delta + i p\cdot\bar{a}$

Now, let's inject the results in 1-loop effective potential:

$$V_{eff}^{(1)}[\bar{A}] = \frac{1}{2i}\,\mathrm{tr}\ln \tilde{D}_{bos}^{-1}\tilde{D}_{bos}^{\bar{A}} - \frac{1}{i}\,\mathrm{tr}\ln \tilde{D}_{gh}^{-1}\tilde{D}_{gh}^{\bar{A}}$$

$$= \frac{1}{2i}\,\mathrm{tr}\sum_{n=1}\frac{(-1)^{n-1}}{n}\underbrace{\left(\begin{array}{c|c} \frac{i(p\cdot\bar{a})\delta - ip\bar{a}}{p^2} & 0 \\ \hline 0 \text{ or } B & 0\end{array}\right)}^{n} - \frac{1}{i}\,\mathrm{tr}\sum_{n=1}\frac{(-1)^{n-1}}{n}\left(\frac{i(p\cdot\bar{a})}{p^2}\right)^n$$

Since $\ldots$ is block-triangular, so $(\ldots)^n$ will be too. We then take the trace of it ⇔ only consider the diagonal. We then set $B = -ip_\lambda\, \epsilon^{\lambda\sigma\rho}\,\bar{a}_\sigma/p^2$ to zero.

$$V_{eff}^{(1)}[\bar{A}] = \sum_{n=1}\frac{(-1)^{n-1}}{n}(i^{n-1})\,p^{-2n}\left(\frac{1}{2}\,\mathrm{tr}\left\{(p\cdot\bar{a}\delta - p\bar{a})^n - (p\cdot\bar{a})^n\right\}\right)$$

Now, notice that $(p\cdot\bar{a}\delta - p\bar{a})^2 = (p\cdot\bar{a})^2\delta - (p\cdot\bar{a})p\bar{a} - p\bar{a}(p\cdot\bar{a}) + p\bar{a}\cdot p\bar{a}$

By recursion, $(p\cdot\bar{a}\delta - p\bar{a})^n = (p\cdot a)^n\delta - p\bar{a}(p\cdot\bar{a})^{n-1}$

We get

$$V_{eff}^{(1)}[\bar{A}] = \sum_{n=1}\frac{(-1)^{n-1}}{n}(i^{n-1})\,p^{-2n}\left(\frac{1}{2}\,\mathrm{tr}\left\{(p\cdot a)^n\delta - p\bar{a}(p\cdot\bar{a})^{n-1}\right\} - (p\cdot\bar{a})^n\right)$$

$$= \sum_{n=1}\frac{(-1)^{n-1}}{n}(i^{n-1})\,p^{-2n}\left(\frac{1}{2}\,\mathrm{tr}_G\left\{3(p\cdot\bar{a})^n - (p\cdot\bar{a})^n\right\} - \mathrm{tr}_G\left\{(p\cdot\bar{a})^n\right\}\right)$$

$= 0$ → The 1-loop effective potential vanishes: there is a cancellation between the gauge bosons and the ghosts.

↳ The coupling constant g is not renormalized perturbatively. This means that the β-function vanishes $\beta(g) = 0$

5.6 Gauge independence and Zinn-Justin eq.

→ When considering a gauge theory, one has a classical gauge invariant action (ex: in YM, $\mathcal{L}$ invariant under
$$\delta_\epsilon A_\mu^a = D_\mu \varepsilon^a = \partial_\mu \varepsilon^a + f_b{}^a{}_c A_\mu^b \varepsilon^c)$$
In order to quantize the theory and perform perturbation expansions, one must fix a gauge. To do so, one introduces a gauge-fixing term in the action and associated ghost fields.
↳ After gauge fixing, the initial local gauge invariance is not manifest anymore.
→ A new global fermionic symmetry, the BRST symmetry, emerges.
$$s A_\mu^a = D_\mu C^a \qquad s C^a = \frac{-1}{2} f_b{}^a{}_c C^b C^c \qquad s \bar{C}^a = B^a \qquad s B^a = 0$$
$$\text{and } s y^i = -C^a (T_a)^i{}_j y^j$$
→ Now, we want to make sure the physical results are independent from the choice of gauge

→ Physical results, such as the S-matrix and the Green function, shouldn't depend on the choice of gauge fixing. Let us focus on Green's functions of product of operators:
$$\langle Vac, out | \Upsilon \prod_i \hat{\mathcal{O}}^i (x_i) | Vac, in \rangle_\Psi$$
$$= \int \mathcal{D}\phi \prod_i \mathcal{O}^i(x_i)\, e^{i/\hbar S_\Psi[\phi, \phi^* = 0]}$$
where Ψ is the gauge-fixing fermion.
We suppose $s \prod_i \hat{\mathcal{O}}^i(x_i) = 0$, operator BRST invariant ⇒ gauge invariant if they don't depend on $C^a, \bar{C}^a, B^a, C_a^*, \bar{C}_a^*$ or B_a^*. since
$$s A_\mu^a = D_\mu C^a \text{ and } \delta_{\varepsilon_0} A_\mu^a = D_\mu \varepsilon^a$$
→ By definition, $S_\Psi[\phi, \phi^* = 0] = S^{inv} - s\Psi$, so that
$$s_\Psi S_\Psi\, S_\Psi[\phi, \phi^* = 0] = s\, S_\Psi[\phi, \phi^* = 0] = 0$$

→ Under a variation $\Psi \mapsto \Psi + \delta\Psi$, one has
$$\langle Vac, out | \Upsilon \prod_i \hat{\mathcal{O}}^i(x_i) | Vac, in \rangle_{\Psi + \delta\Psi} - \langle Vac, out | \Upsilon \prod_i \hat{\mathcal{O}}^i(x_i) | Vac, in \rangle_\Psi$$
Now, recall that $S_\Psi = S^{inv} - s\Psi$ so that
$$S_{\Psi + \delta\Psi} = S^{inv} - s\Psi - s\delta\Psi = S_\Psi - s\delta\Psi$$
We use that $e^{x+\epsilon} = e^x + \epsilon e^x$

We get: $-\frac{i}{\hbar}\int \mathcal{D}\phi \, s\,\delta\phi \, \prod_i \mathcal{O}^i(x_i)\, e^{\frac{i}{\hbar}S_\Psi[\phi,\phi^*=0]}$

Now, $s = \int d^n x \; s\phi^A(x) \frac{\delta^L}{\delta\phi^A(x)}$ so that

$$-\frac{i}{\hbar}\int d^n x \int \mathcal{D}\phi \; s\phi^A(x) \frac{\delta^L}{\delta\phi^A(x)} (\delta\Psi) \prod_i \mathcal{O}^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\phi^*=0]}$$

$$= -\frac{i}{\hbar}\int d^n x \int \mathcal{D}\phi \underbrace{\frac{\delta^L}{\delta\phi^A(x)}\left(s\phi^A(x)\, \delta\Psi \prod_i \mathcal{O}^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\phi^*=0]}\right)}_{} .(-1)^{A(A+1)}$$

$$+\frac{i}{\hbar}\int d^n x \int \mathcal{D}\phi \left(\frac{\delta}{\delta\phi^A(x)} (s\phi^A(x))\, \delta\Psi \prod_i \mathcal{O}^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\phi^*=0]} \right.$$
$$\left. \underset{(-1)^A\cdot(-1)^{A+1}=}{(-)}\, \delta\Psi\, s\underbrace{\left(\prod_i \mathcal{O}^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\phi^*=0]}\right)}_{} \right)$$

$\sim\sim = 0$ by translation invariance of the measure $\mathcal{D}\phi = \mathcal{D}(\phi+\epsilon)$

$\sim\sim = 0$ by BRST invariance of $\prod_i \mathcal{O}^i$ and $S_\Psi[\phi,\phi^*=0]$

We're left with a term $\propto \frac{\delta}{\delta\phi^A}(s\phi^A)$. Now, $\frac{\delta}{\delta\phi^A(x)}\phi^B(y) = \delta^B_A\,\delta(x-y)$

so that we'll get a formal divergence $\delta(0)$ if non vanishing coefficient.

$\rightarrow \frac{\delta(sA^a_\mu(x))}{\delta A^a_\mu(x)} = \delta(0).f_a{}^a{}_c C^c = 0$ $\quad (sA^a_\mu = D_\mu C^a = \partial_\mu C^a + f_b{}^a{}_c A^b_\mu C^c)$

$\frac{\delta(sC^a(x))}{\delta C^a(x)} = -\delta(0)\, f_a{}^a{}_c C^c = 0$ $\quad (sC^a = \frac{-1}{2} f_b{}^a{}_c C^b C^c)$

$\frac{\delta(s\bar{C}^a(x))}{\delta\bar{C}^a(x)} = 0$; $\frac{\delta(sB^a(x))}{\delta B^a(x)} = 0$; $\frac{\delta(sy^i(x))}{\delta y^i(x)} = -\delta(0)\, C^a T_a{}^i{}_i = 0$

↳ There $f_a{}^a{}_b$ vanishes only for compact group with traceless representation.

PROP | We showed that

$$\frac{\delta}{\delta\Psi}\left(\langle Vac, out | T \prod_i \hat{\mathcal{O}}^i(x_i) | Vac, in\rangle\right) = 0$$

→ Physical results (including computation of the β-function) don't depend on the choice of gauge-fixing.

⊙ Ward identities associated to BRST invariance :

→ We started with a (local) gauge invariance, that we break (to make perturbative computation) through gauge-fixing. We then find BRST invariance: $s(S^{inv} - s\Psi) = 0$
We replace a gauge parameter ε^a by a field from the theory C^a.
→ BRST symmetry is fermionic and global.

→ Now, for each global symmetry, there is a conserved current.
(classical: Noether thm ; quantum: Ward id).
Let us find the Ward id. for the BRST symmetry.

→ Consider the change of variable $\phi^A \mapsto \phi^A + \epsilon\, s\phi^A$ in the path integral $Z[J, \phi^*] = \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\left(S_\Psi[\phi, \phi^*] + \int d^n x\, J_A \phi^A\right)\right\}$

where $S_\Psi[\phi, \phi^*] = S^{inv}[\phi] - s\Psi - \int d^n x\, s\phi^A(x)\, \phi^*_A(x)$

We had $0 = \frac{1}{2}(S_\Psi, S_\Psi)_{\phi,\phi^*} = -\int d^n x \frac{\delta^R S_\Psi}{\delta \phi^*_A(x)} \frac{\delta^L S_\Psi}{\delta \phi^A(x)} = \int d^n x\, s\phi^A \frac{\delta^L S_\Psi}{\delta\phi^A(x)}$

$\Leftrightarrow s S_\Psi = 0$ the BRST invariance. Applying the change of variable in Z would give the Ward identities. But let's do it another way.

→ Notice that $\int \mathcal{D}\phi \int d^n x \frac{\delta^L}{\delta\phi^A(x)}\left(s\phi^A(x)\, e^{\frac{i}{\hbar}(S_\Psi + J\phi)}\right) = 0$

$= \int \mathcal{D}\phi \int d^n x\, (-1)^A J_A(x)\, s\phi^A(x)\, e^{i/\hbar (S_\Psi + J\phi)}$ // $1/Z[J, \phi^*]$

$\Leftrightarrow \int d^n x\, (-1)^A J_A(x) \langle s\phi^A \rangle^{J,\phi^*} = 0$

DEF | The Slavnov-Taylor identity is

$$\int d^n x\, (-1)^A J_A(x) \langle s\phi^A(x) \rangle^{J,\phi^*} = 0$$

→ The Slavnov-Taylor id. ensure that physical quantity are gauge independant, and constraint counterterms in the renormalization process.

⊙ Legendre transform and Zinn-Justin equation

→ As in chapter 3, we write the generating functional of connected Green's function as

$$W[J, \tilde{\Phi}^*] = \frac{\hbar}{i} \ln \frac{Z[J, \tilde{\Phi}^*]}{Z[0,0]}$$

One has $\langle s \phi^A \rangle^{J, \tilde{\Phi}^*} = - \frac{\delta^R W}{\delta \tilde{\Phi}^*_A(x)}$ and $- \int d^n x \, (-1)^A J_A(x) \frac{\delta^R W}{\delta \tilde{\Phi}^*_A} = 0$

→ As before, we introduce the classical field

$$\phi^A_{J, \tilde{\Phi}^*}(x) \equiv \frac{\delta^L W[J, \tilde{\Phi}^*]}{\delta J_A(x)}$$

Inverting the relation gives $J_A(x) = J_A^{\phi, \tilde{\Phi}^*}$

→ The effective action is $\Gamma[\phi, \tilde{\Phi}^*] \equiv \left(W[J, \tilde{\Phi}^*] - J\phi \right)\Big|_{J = J^{\phi, \tilde{\Phi}^*}}$

Now, notice that

$$\frac{\delta^R \Gamma}{\delta \tilde{\Phi}^*_A(x)} = \frac{\delta^R W}{\delta \tilde{\Phi}^*_A(x)} + \int d^n y \, \underbrace{\frac{\delta^R W}{\delta J_B(y)}}_{\phi^B(y)\cdot(-1)^B} \frac{\delta^R J_B^{\phi, \tilde{\Phi}^*}}{\delta \tilde{\Phi}^*_A} - \int d^n y \, \frac{\delta^R J_B^{\phi, \tilde{\Phi}^*}}{\delta \tilde{\Phi}^*_A(x)} \cdot \phi^B(y) \, (-1)^{(A+1)B} = \frac{\delta^R W}{\delta \tilde{\Phi}^*_A(x)}$$

and $\frac{\delta^R \Gamma[\phi, \tilde{\Phi}^*]}{\delta \phi^A(x)} = - J_A^{\phi, \tilde{\Phi}^*}(x)$

→ Now, $\int d^n x \, (-1)^A J_A^{\phi, \tilde{\Phi}^*}(x) \, (-1) \frac{\delta^R W}{\delta \tilde{\Phi}^*_A} = 0$

$$\Leftrightarrow \int d^n x \, (-1)^A \frac{\delta^R \Gamma}{\delta \phi^A(x)} \frac{\delta^R W}{\delta \tilde{\Phi}^*_A} = \int d^n x \, (-1)^{A + (A+1)} \frac{\delta^R \Gamma}{\delta \phi^A(x)} \frac{\delta^L W}{\delta \tilde{\Phi}^*_A(x)} = 0$$

PROP | The Zinn-Justin equation reads

$$\frac{1}{2} (\Gamma, \Gamma)_{\phi, \tilde{\Phi}^*} = 0 \qquad // \; \frac{1}{2} (S_\Phi, S_\Phi)_{\phi, \tilde{\Phi}^*} = 0$$

→ Writing the 1-loop effective action $\Gamma = S_\Phi(\phi, \tilde{\Phi}^*) + \hbar \Gamma^{(1)} + \dots$,
one has $(S_\Phi + \hbar \Gamma^{(1)}, S_\Phi + \hbar \Gamma^{(1)}) = 0 + \mathcal{O}(\hbar^2)$

$\Rightarrow (S_\Phi, \Gamma^{(1)}) + (S_\Phi + \Gamma^{(1)}) = 0$

$\Leftrightarrow (S_\Phi, \Gamma^{(1)}) = 0$. Writing $\Gamma^{(1)} = \Gamma^{(1)}_\infty + \Gamma^{(1)}_{finite}$, one has
$(S_\Phi, \Gamma^{(1)}_{div}) = 0$ and $(S_\Phi, \Gamma^{(1)}_{finite}) = 0$

PROP | $\Delta_\Phi \Gamma^{(1)}_\infty = 0 = \Delta_\Phi \Gamma^{(1)}_{finite}$